Journal of Pure and Applied Algebra 14 (1979) 117–129.

TENSOR PRODUCTS OF THEORIES, APPLICATION TO INFINITE LOOP SPACES*

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Communicated by S. MacLane
Received 18 October 1977
Revised 3 May 1978

1. Introduction

In [4] Lawvere formalized the concept of an algebraic theory defined by operations and laws without existential quantifiers. Examples are the theories of monoids, groups, loops, rings, modules etc., whose axioms can be put into the required form; although not the theory of fields. The advantage of this approach lies in the unified treatment of a variety of universal constructions, especially the construction of free objects.

Let $\mathscr{A}$ and $\mathscr{B}$ be algebraic theories, possibly topologized (we recall definitions in Section 2). One frequently is interested in the algebraic structure of $\mathscr{B}$-objects in the category of $\mathscr{A}$-spaces. This is given by the tensor product theory $\mathscr{A} \otimes \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$. For example, group objects in the category of groups are exactly the abelian groups, so that $\mathscr{G} \otimes \mathscr{G}$ is the theory of abelian groups if $\mathscr{G}$ denotes the theory of groups. In general, the structure of $\mathscr{A} \otimes \mathscr{B}$ is far from clear and few examples are known.

We prove the following somewhat surprising result.

1.1. Theorem. *Let $\mathscr{M}$ be the theory of monoids and $\mathscr{A}$ an arbitrary theory having exactly one constant. Then $\mathscr{M} \otimes \mathscr{A}$ is completely determined by the spaces of unary and binary operations in $\mathscr{A}$ (and the structure maps between them).*

* Research partially supported by the National Science Foundation under grants MCS70-01647 and MCS76-23466.

This is immediate from our Main Theorem 4.3, which provides a detailed description of $\mathcal{M} \otimes \mathcal{A}$. Remark 4.7 discusses the situation for other theories $\mathcal{A}$. In Section 5 we specialize to theories that arise from PROPs, in which case we have the more explicit Theorem 5.5. These in turn are applied in Section 6 to infinite loop spaces. We use the terminology of [2].

1.2. Theorem. *Let X be an E-space whose multiplication admits a homotopy inverse (e.g. X numerably contractible and $\pi_0(X)$ a group). Then X is an infinite loop space in the sense that there is a sequence of based E-spaces $X = X_0, X_1, X_2, \ldots$ and homotopy equivalences $X_i \simeq \Omega X_{i+1}$. Moreover, the homotopy equivalences respect the E-structures up to coherent homotopies in the sense of* [2, (4.1)].

This is the analog in our theory of a result of May [5, (3.7)]. It is of importance for the identification of Dyer–Lashof operations in the delooping process and for verifying the May–Thomason axioms [6] for an infinite loop space machine. In [2] we proved only that the homotopy equivalences respect the homotopy monoid structures inherited from the E-structures and the loop additions. The present proof reverts to our original proof sketched in [1, Section 12].

2. Algebraic theories

Here we review the facts we need about theories [2] and fix our notation.

In topological applications we find it convenient throughout to use k-spaces (compactly generated spaces) in the sense of McCord [7], except that we impose no separation axiom. The category of these spaces and maps, which we shall call $\mathcal{T}\!op$, was expounded in [10]. All products, function spaces, subspaces etc. are taken inside this category. In this manner all constructed spaces automatically receive a natural topology.

Let $\mathcal{N}$ be the category with objects $0, 1, 2, \ldots$, with $\mathcal{N}(m, n)$ the set of all functions $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$, and with usual composition. (The object 0 corresponds to the empty set. Thus $\mathcal{N}$ is equivalent to the category of finite sets.) In the dual category $\mathcal{N}^{\mathrm{op}}$ we write $\sigma^* \in \mathcal{N}^{\mathrm{op}}(n, m)$ for the morphism dual to $\sigma \in \mathcal{N}(m, n)$, and call σ^* a *set operation*. In $\mathcal{N}^{\mathrm{op}}$, the object n is the product of n copies of the object 1 by means of the n *projection* morphisms $p_i \in \mathcal{N}^{\mathrm{op}}(n, 1)$ dual to the functions taking 1 to i $(1 \leq i \leq n)$.

Suppose $X : \mathcal{N}^{\mathrm{op}} \to \mathcal{T}\!op$ is a product-preserving functor in the strict sense that $Xn = (X1)^n$ for each n. We call $X1$ the *underlying space*; it determines X. By abuse of notation we write X for $X1$ and σ^* for $X\sigma^*$. For any set operation $\sigma^* \in \mathcal{N}^{\mathrm{op}}(n, m)$ we must have

$$\sigma^*(x_1, x_2, \ldots, x_n) = (x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma m}) \quad (x_i \in X) \tag{2.1}$$

since $p_i \circ \sigma^* = p_{\sigma i}$ in $\mathcal{N}^{\mathrm{op}}$. In particular, $p_i : X^n \to X$ is indeed the projection to the ith factor, and π^* permutes the factors of X^n if π is a permutation of $\{1, 2, \ldots, n\}$. Other important set operations are the *identity* $\mathrm{id}_n : X^n = X^n$, $\mathrm{id} = \mathrm{id}_1$, and the n-fold *diagonal* $\Delta_n : X \to X^n$.

The central idea is to enrich X by enlarging the category $\mathcal{N}^{\mathrm{op}}$. An *algebraic theory* is a category $\mathscr{A}$ with objects 0, 1, 2, ..., equipped with a functor (suppressed from our notation) $\mathcal{N}^{\mathrm{op}} \to \mathscr{A}$ that preserves objects and products, so that we are in effect given isomorphisms

$$\mathscr{A}(m, n) = \mathscr{A}(m, 1)^n. \tag{2.2}$$

We lose no real generality if we assume these isomorphisms to be identities (or, equivalently, suppress them from the notation), so that $\mathscr{A}$ is completely determined by the sets $\mathscr{A}(m, 1)$ and the compositions $\mathscr{A}(n, 1) \times \mathscr{A}(m, n) \to \mathscr{A}(m, 1)$. The elements of $\mathscr{A}(m, 1)$ are called the *m-ary operations*; in particular, *binary* operations ($m = 2$), *unary* operations ($m = 1$) and *constants* ($m = 0$). As in (2.1) we have for any set operation $\sigma^* \in \mathcal{N}^{\mathrm{op}}(n,m)$

$$\sigma^* \circ (a_1, a_2, \ldots, a_n) = (a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma m}), \quad a_i \in \mathscr{A}(k, 1). \tag{2.3}$$

Given $a \in \mathscr{A}(m, n)$ and $b \in \mathscr{A}(k, l)$, their *product* $a \times b \in \mathscr{A}(m+k, n+l)$ is particularly useful. For the product of operations $b_i \in \mathscr{A}(r_i, 1)$, (2.3) reduces to

$$\sigma^* \circ (b_1 \times b_2 \times \cdots \times b_n) = (b_{\sigma 1} \times b_{\sigma 2} \times \cdots \times b_{\sigma m}) \circ \bar{\sigma}^*, \tag{2.4}$$

where $\bar{\sigma}^* \in \mathcal{N}^{\mathrm{op}}(\sum_i r_i, \sum_j r_{\sigma j})$ is a rather messy set operation depending only on σ and the r_i. This allows us to pass set operations to the right. In our applications we need *topological* algebraic theories (to take care of higher homotopies etc.), in which the morphism sets $\mathscr{A}(m, n)$ are topologized, the isomorphisms (2.2) are homeomorphisms, and composition and functors are required to be continuous. A *theory functor* $\mathscr{A} \to \mathscr{B}$ is a functor that preserves the subcategory $\mathcal{N}^{\mathrm{op}}$.

An *$\mathscr{A}$-space* is a product-preserving functor $X : \mathscr{A} \to \mathscr{T}\!op$, and a *homomorphism* of $\mathscr{A}$-spaces is a natural transformation between such functors. We again write Xb as b for operations b in $\mathscr{A}$. More generally, we may replace $\mathscr{T}\!op$ by any category that admits finite products, to define $\mathscr{A}$-objects in that category. In particular, the space $\mathscr{A}(n, 1)$ is itself an $\mathscr{A}$-space under composition (with the help of (2.2)); indeed, the key observation [4] is that $\mathscr{A}(n, 1)$ is the *free* $\mathscr{A}$-space on the set $\{p_1, p_2, \ldots, p_n\}$ of projections (essentially because $\mathrm{id}_n = (p_1, p_2, \ldots, p_n)$). Thus *to know the free $\mathscr{A}$-space on any finite set S is to know the theory $\mathscr{A}$.*

The category of $\mathscr{A}$-spaces and homomorphisms admits obvious products. If X and Y are $\mathscr{A}$-spaces, so is $X \times Y$ under the coordinatewise operation of $c \in \mathscr{A}(n, 1)$ given by

$$c((x_1, y_1), \ldots, (x_n, y_n)) = (c(x_1, \ldots, x_n), c(y_1, \ldots, y_n)).$$

By repetition, the powers X^k of X become $\mathscr{A}$-spaces in which $a \in \mathscr{A}(m, n)$ operates by

$$(X^k)^m \xrightarrow[\tau^*_{k,m}]{} (X^m)^k \xrightarrow[a\times\cdots\times a]{} (X^n)^k \xrightarrow[\tau^*_{n,k}]{} (X^k)^n, \tag{2.5}$$

where $\tau^*_{k,m} \in \mathcal{N}^{\mathrm{op}}(km, km)$ is the appropriate *shuffle* set operation (in symbols, $\tau_{k,m}((i-1)m+j)=(j-1)k+i$ for $1 \leq i \leq k$ and $1 \leq j \leq m$).

One may therefore consider $\mathcal{B}$-objects X in the category of $\mathcal{A}$-spaces. Each operation $b \in \mathcal{B}(k, l)$ on X is a homomorphism of $\mathcal{A}$-spaces, which means that the *interchange* diagram (e.g. [1, Section 4] or [2, p. 42])

$$\begin{array}{ccc} (X^k)^m \cong (X^m)^k & \xrightarrow[a\times a\times\cdots\times a]{} & (X^n)^k \cong (X^k)^n \\ \Big\downarrow{\scriptstyle b\times b\times\cdots\times b} & & \Big\downarrow{\scriptstyle b\times b\times\cdots\times b} \\ (X^l)^m \cong (X^m)^l & \xrightarrow[a\times a\times\cdots\times a]{} & (X^n)^l \cong (X^l)^n \end{array} \tag{2.6}$$

commutes for all $a \in \mathcal{A}(m, n)$. In other words, X is simultaneously an $\mathcal{A}$-space and a $\mathcal{B}$-space in such a way that the actions interchange, a completely symmetrical situation. There is a natural way to define a theory $\mathcal{A} \otimes \mathcal{B}$ that acts on such spaces; it comes equipped with functors $\mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $\mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ that enjoy the obvious universal property. One can construct $\mathcal{A} \otimes \mathcal{B}$ as generated under composition by the images of $\mathcal{A}$ and $\mathcal{B}$, with the relations (2.6) imposed for all a in $\mathcal{A}$ and b in $\mathcal{B}$. Fortunately, it is not necessary to check (2.6) for *all* pairs (a, b). We need only check for each coordinate of a and b, which is trivial if a or b is a set operation. It holds for $(a' \circ a'', b)$ if it holds for (a', b) and (a'', b), and similarly for $(a, b' \circ b'')$. In general this still leads to a severe word (or tree) problem. As mentioned in the Introduction hardly anything is known about tensor products of theories. We state two results whose proofs can be found in [8].

2.7. Proposition. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are (topologized) algebraic theories each having at least one constant. Then $\mathcal{A} \otimes \mathcal{B}$ has exactly one constant.*

2.8. Proposition. *Let $\mathcal{M}$ be the theory of monoids and $\mathcal{CM}$ the theory of commutative monoids. Then $\mathcal{CM} = \mathcal{M} \otimes \mathcal{M} \otimes \cdots \otimes \mathcal{M}$* ($n$ times, any $n \geq 2$).

3. Relations in $\mathcal{M} \otimes \mathcal{A}$-spaces.

Let X be *any* $\mathcal{M} \otimes \mathcal{A}$-space, where $\mathcal{M}$ is the theory of monoids and $\mathcal{A}$ is any theory with exactly one constant, $o \in \mathcal{A}(0, 1)$. We study the relations that hold in X. For convenience we denote the monoid multiplication in X (coming from the $\mathcal{M}$-action) by $*$ (as in Fortran) and its identity element by e. We adopt the convention that $*$ is evaluated after the other operations, to save parentheses.

3.1. Definition. Given $u \in \mathscr{A}(n, 1)$, we define the ith *axis* $u(i)$ of u (for $1 \leq i \leq n$) as

$$u(i) = u \circ (o \times \cdots \times \mathrm{id} \times \cdots \times o) \in \mathscr{A}(1, 1), \quad \text{(id in } i\text{th place).}$$

The most important property is that the action of u on X depends only on the actions of its axes.

3.2. Lemma. *In any $\mathscr{M} \otimes \mathscr{A}$-space X the following relations hold, where x_i, x, $y \in X$, $a \in \mathscr{A}(1, 1)$ and $v \in \mathscr{A}(2, 1)$:*

$$\begin{aligned}
&\text{(a)}\ u(x_1, x_2, \ldots, x_n) = u(1)x_1 * u(2)x_2 * \cdots * u(n)x_n, \\
&\text{(b)}\ u(i)x * u(j)y = u(j)y * u(i)x \quad \text{if } i \neq j, \\
&\text{(c)}\ o = e, \\
&\text{(d)}\ a(x * y) = ax * ay, \\
&\text{(e)}\ ae = e, \\
&\text{(f)}\ (v \circ \Delta_2)x = v(1)x * v(2)x, \\
&\text{(g)}\ (o \circ \Delta_0)x = e, \\
&\text{(h)}\ (u \circ \Delta_n)x = u(1)x * u(2)x * \cdots * u(n)x.
\end{aligned} \tag{3.3}$$

Proof. Interchange of $o \in \mathscr{A}(0, 1)$ with $e \in \mathscr{M}(0, 1)$ gives (c) directly. Also, (d) and (e) are exactly the interchange relations of a with $*$ and e. Then interchange of $u \in \mathscr{A}(n, 1)$ with the n-fold multiplication $\lambda_n \in \mathscr{M}(n, 1)$ yields, by introducing extra factors e which we change to o by (c),

$$\begin{aligned}
u(x_1, x_2, \ldots, x_n) &= u(\lambda_n(x_1, \ldots, e), \lambda_n(e, x_2, \ldots, e), \ldots, \lambda_n(e, \ldots, x_n)) \\
&= \lambda_n(u(x_1, \ldots, e), u(e, x_2, \ldots, e), \ldots, u(e, \ldots, x_n)) \\
&= u(1)x_1 * u(2)x_2 * \cdots * u(n)x_n,
\end{aligned}$$

as required.

Interchange of u with $*$ gives in general

$$u(x_1 * y_1, \ldots, x_n * y_n) = u(x_1, \ldots, x_n) * u(y_1, \ldots, y_n).$$

If we take $x_j = y$, $y_i = x$, and other entries e we find (assuming for convenience that $i < j$)

$$u(e, \ldots, x, \ldots, y, \ldots, e) = u(j)y * u(i)x.$$

On the other hand, (a) and (e) give $u(i)x * u(j)y$ instead; hence (b).

Relation (h) is immediate from (a), since $\Delta_n x = (x, x, \ldots, x)$. Finally, (f) and (g) are important special cases of (h).

4. Free $\mathscr{M} \otimes \mathscr{A}$-spaces

We propose to construct the *free $\mathscr{M} \otimes \mathscr{A}$-space* on a given set (or space) S using only $\mathscr{A}(1, 1)$ and $\mathscr{A}(2, 1)$, assuming that $\mathscr{A}$ has exactly one constant. As indicated in

Section 2, this will imply the structure of $\mathcal{M} \otimes \mathcal{A}$. Relations (3.3) must hold in the space we construct and they therefore serve as a guide for its construction.

First we introduce a relation on the space $\mathcal{A}(1, 1)$.

4.1. Definition. Given two elements $a, b \in \mathcal{A}(1, 1)$, we write $a \perp b$ if and only if there exists a binary operation $u \in \mathcal{A}(2, 1)$ such that $a = u(1)$ and $b = u(2)$.

4.2. Lemma. *The relation $\perp$ has the following properties*:

(a) (symmetry) *$a \perp b$ implies $b \perp a$,*

(b) *$a \perp b$ implies $c \circ a \perp c \circ b$ for any $c \in \mathcal{A}(1, 1)$,*

(c) *$a \perp b$ implies $a \circ c \perp b \circ d$ for any $c, d \in \mathcal{A}(1, 1)$,*

(d) *$u(i) \perp u(j)$ whenever $u \in \mathcal{A}(n, 1)$ and $i \neq j$.*

Proof. Suppose throughout that $a = v(1)$ and $b = v(2)$ for some $v \in \mathcal{A}(2, 1)$. Take $w = v \circ \tau^*$, where $\tau \in \mathcal{N}(2, 2)$ is the transposition; then $w(1) = v(2) = b$ and $w(2) = v(1) = a$ shows that $b \perp a$, which is (a). For (b) we take $w = c \circ v \in \mathcal{A}(2, 1)$, so that $w(1) = c \circ a$ and $w(2) = c \circ b$. For (c) we take $w = v \circ (c \times d)$, so that $w(1) = a \circ c$ and $w(2) = b \circ d$. In (d) we may assume (by (a)) that $i < j$ and take $w = u \circ (o \times \cdots \times \mathrm{id} \times \cdots \times \mathrm{id} \times \cdots \times o) \in \mathcal{A}(2, 1)$ (with id in ith and jth places only), so that $w(1) = u(i)$ and $w(2) = u(j)$.

We can now describe the free $\mathcal{M} \otimes \mathcal{A}$-space we seek.

4.3. Theorem. *Let $\mathcal{M}$ be the theory of monoids and $\mathcal{A}$ any theory with exactly one constant, o. Given any set S, let F be the monoid with identity element e and multiplication $*$ generated by the symbols as for $a \in \mathcal{A}(1, 1)$ and $s \in S$, subject to the relations*

$$
\begin{aligned}
&\text{(a)}\quad as * bt = bt * as \text{ whenever } a \perp b,\ (s, t \in S),\\
&\text{(b)}\quad (u \circ \Delta_2)s = u(1)s * u(2)s, \quad (u \in \mathcal{A}(2, 1), s \in S),\\
&\text{(c)}\quad (o \circ \Delta_0)s = e.
\end{aligned}
\tag{4.4}
$$

Then the relations (3.3) *determine a unique $\mathcal{M} \otimes \mathcal{A}$-structure on F which makes F the free $\mathcal{M} \otimes \mathcal{A}$-space on the set S, where S is included in F as the set of elements* id s.

As in Section 2, we deduce the structure of $\mathcal{M} \otimes \mathcal{A}$ by taking S as the set of n elements $\{p_1, p_2, \ldots, p_n\}$, for all $n \geq 0$.

4.5. Corollary. *Every operation in $(\mathcal{M} \otimes \mathcal{A})(n, 1)$ can be written in the form $\lambda_k \circ (a_1 \times a_2 \times \cdots \times a_k) \circ \sigma^*$, where $a_i \in \mathcal{A}(1, 1)$ and $\sigma^* \in \mathcal{N}^{\mathrm{op}}(n, k)$ is some set operation.*

The precise description of the relations holding in $\mathcal{M} \otimes \mathcal{A}$ is left as an exercise.

In Theorem 4.3 we have so far defined a monoid F. We have to make the operations in $\mathcal{A}$ act on F in such a way that F becomes a $\mathcal{M} \otimes \mathcal{A}$-space.

4.6. Lemma. *For each $c \in \mathscr{A}(1, 1)$ there is a unique monoid homomorphism $c : F \to F$ satisfying $c(as) = (c \circ a)s$ on the generators as. If we then define the action of any $u \in \mathscr{A}(n, 1)$ on F by* (3.3a), *all the relations* (3.3) *become valid in F.*

Proof. Throughout this proof, relation (x) refers to $(3.3x)$. First we have to check that the proposed homomorphism c respects the relations (4.4) used to define the monoid F. By (4.2b) and (4.4a), $(c \circ a)s * (c \circ b)t = (c \circ b)t * (c \circ a)s$ whenever $a \perp b$, which takes care of (4.4a). For (4.4b), take $w = c \circ u \in \mathscr{A}(2, 1)$, so that $w \circ \Delta_2 = c \circ (u \circ \Delta_2)$ and $w(i) = c \circ u(i)$ in $\mathscr{A}$. Then (4.4b) applied to w gives $(c \circ (u \circ \Delta_2))s = (c \circ u(1))s * (c \circ u(2))s$ in F, which takes care of (4.4b). Since $c \circ (o \circ \Delta_0) = (c \circ o) \circ \Delta_0 = o \circ \Delta_0$ in $\mathscr{A}$ (there being only one constant in $\mathscr{A}$), (4.4c) is preserved. Thus the homomorphism c can be defined, and relations (d) and (e) hold.

Then (a) holds by definition, and if $n = 1$ this definition is consistent with what we already have, since in this case $u(1) = u$. In particular, when $n = 0$ we interpret the empty product as e, which yields (c).

To verify (b), take two general elements $x = a_1 s_1 * \cdots * a_k s_k$ and $y = b_1 t_1 * \cdots * b_m t_m$ of F. By definition,

$$u(i)x = (u(i) \circ a_1)s_1 * \cdots * (u(i) \circ a_k)s_k,$$

$$u(j)y = (u(j) \circ b_1)t_1 * \cdots * (u(j) \circ b_m)t_m.$$

In view of parts (c) and (d) of (4.2) and relation (4.4a), each factor of $u(i)x$ commutes with each factor of $u(j)y$ since $i \neq j$, which yields (b).

By (b), (d) and (e), each side of (f) is (for fixed v) a homomorphism in x, so that we need check equality only on the generators, $(v \circ \Delta_2)as = v(1)as * v(2)as$. This follows by applying (4.4b) with $u = v \circ (a \times a) \in \mathscr{A}(2, 1)$.

The degenerate operation $o \circ \Delta_0 \in \mathscr{A}(1, 1)$ acts on the generators of F by $(o \circ \Delta_0)as = (o \circ (\Delta_0 \circ a))s = (o \circ \Delta_0)s = e$ by (4.4c). Since this operation is a homomorphism, we deduce (g).

Finally, we prove (h) by induction on n. We already have the special cases $n = 2$ and $n = 0$, and for $n = 1$ it is trivial. Therefore assume $n > 2$ and that (h) holds for $n - 1$. Write $v = u \circ (\Delta_{n-1} \times \mathrm{id}) \in \mathscr{A}(2, 1)$ and $w = u \circ (\mathrm{id}_{n-1} \times o) \in \mathscr{A}(n-1, 1)$, so that $v(1) = w \circ \Delta_{n-1}$, $v(2) = u(n)$, and $w(i) = u(i)$ for $i < n$. In $\mathscr{A}$ we have $u \circ \Delta_n = u \circ (\Delta_{n-1} \times \mathrm{id}) \circ \Delta_2 = v \circ \Delta_2$. Therefore by induction we have

$$\begin{aligned}(u \circ \Delta_n)x &= (v \circ \Delta_2)x \\ &= (w \circ \Delta_{n-1})x * u(n)x \quad \text{(by the case } n = 2\text{)} \\ &= u(1)x * u(2)x * \cdots * u(n-1)x * u(n)x.\end{aligned}$$

Proof of Theorem 4.3. We have constructed F as a monoid or $\mathscr{M}$-space and defined an action of each $u \in \mathscr{A}(n, 1)$ on F. We have to verify that these actions *do* make F an $\mathscr{A}$-space, that the $\mathscr{A}$-action interchanges with the $\mathscr{M}$-action, and that F is free on S.

To check the interchange is to check that the action of each $u \in \mathscr{A}(n, 1)$ is a monoid homomorphism, which is clear from (3.3a,b,d,e).

The hardest part is to check that F is in fact an $\mathscr{A}$-space, that $\mathscr{A} \to \mathscr{T}\!op$ is a functor. Clearly the identity operation id acts correctly. We have to check that $u(vz) = wz$, where $w = u \circ v$ in $\mathscr{A}$ and $z \in F^m$. The only case we need consider is when $u \in \mathscr{A}(n, 1)$ and $v = (v_1, v_2, \dots, v_n) \in \mathscr{A}(m, n)$, where $v_i \in \mathscr{A}(m, 1)$. Since both sides define monoid homomorphisms $F^m \to F$, we need check only the special case $z = (e, \dots, x, \dots, e)$ with $x \in F$ in the kth place and e elsewhere. Then $wz = w(k)x$ and $v_i z = v_i(k)x$, so that

$$\begin{aligned} u(vz) &= u(v_1(k)x, \dots, v_n(k)x) \\ &= u(1)v_1(k)x * \cdots * u(n)v_n(k)x. \end{aligned}$$

On the other hand, working in $\mathscr{A}$, we have

$$\begin{aligned} w(k) &= u \circ (v_1, \dots, v_n) \circ (o \times \cdots \times \mathrm{id} \times \cdots \times o) \quad \text{(id in } k\text{th place)} \\ &= u \circ (v_1(k), \dots, v_n(k)) \\ &= u \circ (v_1(k) \times \cdots \times v_n(k)) \circ \Delta_n. \end{aligned}$$

If we apply (3.3h) to $u \circ (v_1(k) \times \cdots \times v_n(k)) \in \mathscr{A}(n, 1)$ we obtain

$$w(k)x = u(1)v_1(k)x * \cdots * u(n)v_n(k)x = u(vz).$$

It is now clear how the action of $\mathscr{A}$ on F was forced on us by the relations (3.3), which must hold in any $\mathscr{M} \otimes \mathscr{A}$-space.

Finally, F is indeed the *free* $\mathscr{M} \otimes \mathscr{A}$-space on S. Given any $\mathscr{M} \otimes \mathscr{A}$-space Y and map $f: S \to Y$, it is clear from (3.3) that the unique $\mathscr{M} \otimes \mathscr{A}$-homomorphism $g: F \to Y$ extending f is given by

$$g(a_1 s_1 * \cdots * a_n s_n) = a_1 f s_1 * \cdots * a_n f s_n.$$

4.7. Remark. The uniqueness of the constant o is essential to our method. If $\mathscr{A}$ has no constants at all, examples show that the structure of $\mathscr{M} \otimes \mathscr{A}$ becomes enormously more complicated. However, if $\mathscr{A}$ has more than one constant, it is not hard to see that there is a universal quotient theory $\mathscr{A}'$ of $\mathscr{A}$ with exactly one constant, for which it follows (compare Proposition 2.7) that $\mathscr{M} \otimes \mathscr{A} = \mathscr{M} \otimes \mathscr{A}'$. Indeed, we can describe $\mathscr{A}'$ directly. On each set $\mathscr{A}(n, 1)$ introduce the symmetric relation R by aRb if and only if there exist $c \in \mathscr{A}(n+1, 1)$ and constants o' and o'' such that $a = c \circ (\mathrm{id}_n \times o')$ and $b = c \circ (\mathrm{id} \times o'')$. Then $\mathscr{A}'(n, 1)$ is the quotient space of $\mathscr{A}(n, 1)$ by the smallest equivalence relation containing R. Hence the generalization of Theorems 4.3 and 1.1 to the case when $\mathscr{A}$ has more than one constant.

5. The special case of PROPs.

From now on we restrict attention to theories of a special kind, namely those with PROPs. For these the Main Theorem 4.3 simplifies usefully. We recall what we need from [1, Section 5].

In the theory $\mathscr{CM}$ of commutative monoids, one is interested primarily in the n-fold multiplications $\lambda_n \in \mathscr{CM}(n, 1)$ for $n \geq 0$ and the permutation operations π^*. The space $\mathscr{CM}(n, 1)$ is inconveniently large, being the free commutative monoid on n generators $p_1, p_2, \ldots, p_n$. However, its general element can be written

$$\begin{aligned} u &= m_1p_1 + m_2p_2 + \cdots + m_np_n \\ &= p_1 + \cdots + p_1 + p_2 + \cdots + p_2 + \cdots + p_n + \cdots + p_n \quad (m_i \text{ copies of } p_i) \\ &= \lambda_m \circ (\Delta_{m_1} \times \Delta_{m_2} \times \cdots \times \Delta_{m_n}) \end{aligned} \tag{5.1}$$

where $m = \sum_i m_i$. More generally, any element of $\mathscr{CM}(n, k)$ has the form $(\lambda_{n_1} \times \lambda_{n_2} \times \cdots \times \lambda_{n_k}) \circ \sigma^*$ for some numbers n_i and set operation σ^*. One therefore introduces the smaller more manageable subcategory $\mathscr{CM}^*$ of $\mathscr{CM}$ consisting of all operations of the form $(\lambda_{n_1} \times \lambda_{n_2} \times \cdots \times \lambda_{n_k}) \circ \pi^*$ with π a *permutation*; so that $\mathscr{CM}^*(n, 1)$ consists of the one point λ_n. (As an abstract category, $\mathscr{CM}^*$ is a copy of the category $\mathscr{N}$ of Section 2.) This is the first and trivial example of a PROP.

Now suppose that $\mathscr{A}$ is a theory over $\mathscr{CM}$, that is, equipped with a theory functor $\mathscr{A} \to \mathscr{CM}$. We define the subcategory $\mathscr{A}^*$ of $\mathscr{A}$ as the inverse image of $\mathscr{CM}^*$. The only set operations remaining in $\mathscr{A}$ are the permutation operations π^*. Although the object n is no longer the categorical product in $\mathscr{A}^*$ of n copies of the object 1, we do have $u \times v$ in $\mathscr{A}^*$ whenever u and v are in $\mathscr{A}^*$. We ask whether (5.1) carries over to $\mathscr{A}$.

5.2. Definition. We call $\mathscr{A}^*$ the PROP associated to the theory $\mathscr{A}$ if and only if every operation $u \in \mathscr{A}(n, 1)$ has the form $v \circ (\Delta_{m_1} \times \Delta_{m_2} \times \cdots \times \Delta_{m_n})$ with $v \in \mathscr{A}^*(m, 1)$ where $m = \sum_i m_i$, uniquely apart from replacing v by $v \circ \pi^*$ for permutations $\pi \in \mathscr{S}_{m_1} \times \mathscr{S}_{m_2} \times \cdots \times \mathscr{S}_{m_n} \subset \mathscr{S}_m$, and if $\mathscr{A}(n, 1)$ has the corresponding quotient topology.

It follows that every operation in $\mathscr{A}(n, k)$ has the form $(u_1 \times u_2 \times \cdots \times u_k) \circ \sigma^*$ for some $u_i \in \mathscr{A}^*(n_i, 1)$ and set operation σ^*.

[From our present point of view, we start with a suitable theory $\mathscr{A}$ over $\mathscr{CM}$ and find a PROP $\mathscr{A}^*$ inside it from which $\mathscr{A}$ can be readily reconstructed. Conversely, given an abstract PROP $\mathscr{A}^*$ (that is, a category equipped with objects 0, 1, 2, ... and a functor $\times : \mathscr{A}^* \times \mathscr{A}^* \to \mathscr{A}^*$ and permutation operations $\pi^* \in \mathscr{A}^*(n, n)$ satisfying appropriate axioms, notably (2.4)), one can construct the theory $\mathscr{A}$ over $\mathscr{CM}$ containing it.]

As an example we have the PROP $\mathcal{M}^*$ of monoids, where $\mathcal{M}$ is viewed as a theory over $\mathcal{CM}$ by means of the abelianization theory functor $\mathcal{M} \to \mathcal{CM}$. The space $\mathcal{M}^*(n, 1)$ consists of the $n!$ distinct operations $\lambda_n \circ \pi^*$.

If theories $\mathcal{A}$ and $\mathcal{B}$ contain PROPs, it is not difficult to see that their tensor product $\mathcal{A} \otimes \mathcal{B}$ does also (essentially because the shuffles used in (2.6) are permutations, or because $\mathcal{CM} \otimes \mathcal{CM} = \mathcal{CM}$); we therefore *define* the tensor product of PROPs $\mathcal{A}^* \otimes \mathcal{B}^*$ as $(\mathcal{A} \otimes \mathcal{B})^*$. As promised, the structure of $\mathcal{M}^* \otimes \mathcal{A}^*$ is significantly simpler.

5.3. Theorem. *Let $\mathcal{M}$ be the theory of monoids and $\mathcal{A}$ a theory that contains a* PROP *$\mathcal{A}^*$ having exactly one constant o. Then the free $\mathcal{M} \otimes \mathcal{A}$-space G on a given set S is the monoid with generators as with $a \in \mathcal{A}^*(1, 1)$ and $s \in S$, subject to the relations*

$$as * bt = bt * as \text{ whenever } a \perp b \text{ in } \mathcal{A}^*(1, 1), \quad (s, t \in S). \tag{5.4}$$

Proof. We deduce this from Theorem 4.3. (Alternatively, we could imitate the proof of Theorem 4.3 with some changes.)

We have to compare the monoid G with the monoid F constructed in Theorem 4.3. There is an obvious monoid homomorphism $\varphi : G \to F$, because the generators and relations for G are subsets of those for F. We plan to construct the inverse homomorphism $\theta : F \to G$.

Suppose the element $a \in \mathcal{A}(1, 1)$ lies over the element $np_1 \in \mathcal{CM}(1, 1)$ (by means of $\mathcal{A}(1, 1) \to \mathcal{CM}(1, 1)$). By Definition 5.2 we can therefore write $a = u \circ \Delta_n$, with $u \in \mathcal{A}^*(n, 1)$. We define

$$\theta(as) = u(1)s * u(2) * \cdots * u(n)s,$$

as suggested by (3.3h). Any other choice of u has the form $u \circ \pi^*$ with $\pi \in \mathcal{S}_n$; but this would merely permute the factors of $\theta(as)$, which is harmless because they all commute in G by (4.2d) and (5.4). So $\theta(as)$ is well defined.

We have to check that θ respects the relations (4.4). Suppose $a = u(1)$ and $b = u(2)$ with $u \in \mathcal{A}(2, 1)$. Suppose u lies over $kp_1 + mp_2 \in \mathcal{CM}(2, 1)$; then by Definition 5.2, $u = v \circ (\Delta_k \times \Delta_m)$ for some $v \in \mathcal{A}^*(k+m, 1)$. We deduce that

$$a = u(1) = v \circ (\Delta_k \times o \times \cdots \times o) = v \circ (\mathrm{id}_k \times o \times \cdots \times o) \circ \Delta_k,$$

$$b = u(2) = v \circ (o \times \cdots \times o \times \Delta_m) = v \circ (o \times \cdots \times o \times \mathrm{id}_m) \circ \Delta_m.$$

By the definition of θ,

$$\theta(as) = v(1)s * v(2)s * \cdots * v(k)s,$$

$$\theta(bt) = v(k+1)t * \cdots * v(k+m)t.$$

These commute in G by (5.4) and (4.2d), which takes care of (4.4a). Further, $u \circ \Delta_2 = v \circ (\Delta_k \times \Delta_m) \circ \Delta_2 = v \circ \Delta_{k+m}$ in $\mathcal{A}$, so that

$$\theta((u \circ \Delta_2)s) = v(1)s * v(2)s * \cdots * v(k+m)s = \theta(as) * \theta(bs),$$

which takes care of (4.6b). Finally, our definition of θ gives $\theta((o \circ \Delta_0)s) = e$ directly, for (4.4c). So θ does extend to a monoid homomorphism $\theta: F \to G$. On the generators, we obviously have $\theta\varphi(as) = as$ if $a \in \mathscr{A}^*(1, 1)$, and (3.3h) shows that $\varphi\theta(as) = as$ in F for $a \in \mathscr{A}(1, 1)$. *So* θ and φ are inverse homorphisms, and G is therefore isomorphic to F.

Remark. This result is more useful than Theorem 4.3 because we can solve the word problem in the monoid G very easily. Explicitly, $a_1s_1 * \cdots * a_ns_n = b_1t_1 * \cdots * b_mt_m$ if and only if $m = n$ and there exists a permutation $\pi \in \mathscr{S}_n$ such that $b_i = a_{\pi i}$ and $t_i = s_{\pi i}$ for all i, and $a_i \perp a_j$ whenever $i < j$ and $\pi^{-1}i > \pi^{-1}j$. No such simple condition is available for Corollary 4.5.

We again deduce the structure of $\mathscr{M} \otimes \mathscr{A}$ by taking S as the set $\{p_1, p_2, \ldots, p_n\}$ of n projections, for $n \geq 0$. The element $ap_i \in (\mathscr{M} \otimes \mathscr{A})(n, 1)$ clearly lies over $p_i \in \mathscr{CM}(n, 1)$. The PROP $\mathscr{M}^* \otimes \mathscr{A}^*$ consists of those elements lying over $\mathscr{CM}^*$, which are therefore those of the form $a_{\pi 1}p_{\pi 1} * a_{\pi 2}p_{\pi 2} * \cdots * a_{\pi n}p_{\pi n}$ $(a_i \in \mathscr{A}^*(1, 1), \pi \in \mathscr{S}_n)$.

5.5. Theorem. *In the* PROP $\mathscr{M}^* \otimes \mathscr{A}^*$, *the space* $(\mathscr{M}^* \otimes \mathscr{A}^*)(n, 1)$ *is a quotient space of* $\mathscr{S}_n \times \mathscr{A}^*(1, 1)^n$. *Its elements have the form* $\lambda_n \circ \pi^* \circ (a_1 \times a_2 \times \cdots \times a_n)$, *subject to the relations*

$$\lambda_n \circ \pi^* \circ (a_1 \times a_2 \times \cdots \times a_n) = \lambda_n \circ \rho^* \circ (a_1 \times a_2 \times \cdots \times a_n) \tag{5.6}$$

if and only if $\pi^{-1}i < \pi^{-1}j$ *and* $\rho^{-1}i > \rho^{-1}j$ *imply* $a_i \perp a_j$.

Hence (2.8), because if $\mathscr{A} = \mathscr{M}$ or $\mathscr{CM}$, $\mathscr{A}^*(1, 1)$ consists of only the operation id, and id $\perp$ id.

Remark. If $\mathscr{A}^*$ happens to contain more than one constant, we can form a quotient PROP $\mathscr{A}'^*$ with exactly one constant, for which $\mathscr{M}^* \otimes \mathscr{A}^* = \mathscr{M}^* \otimes \mathscr{A}'^*$, just as we did in Remark 4.7 for theories. On the other hand, if $\mathscr{A}^*$ has no constants at all, examples show that the structure of $\mathscr{M}^* \otimes \mathscr{A}^*$ *does* depend on the spaces $\mathscr{A}^*(n, 1)$ for all n.

5.7. Corollary. *The action of* $\mathscr{S}_n$ *on* $(\mathscr{M}^* \otimes \mathscr{A}^*)(n, 1)$, *for* $n > 1$, *is free if and only if* $a \perp a$ *is false for all* $a \in \mathscr{A}^*(1, 1)$.

Proof. If $a \perp a$, we clearly have in $\mathscr{M}^* \otimes \mathscr{A}^*$ from (5.6)

$$\lambda_n \circ \pi^* \circ (a \times a \times \cdots \times a) = \lambda_n \circ (a \times a \times \cdots \times a).$$

From (2.4) we also have

$$\lambda_n \circ \pi^* \circ (a \times a \times \cdots \times a) = \lambda_n \circ (a \times a \times \cdots \times a) \circ \pi^*.$$

These show that $\lambda_n \circ (a \times a \times \cdots \times a)$ is in fact fixed under the action of $\mathscr{S}_n$.

Conversely, suppose the action is not free, that

$$\lambda_n \circ \pi^* \circ (a_1 \times \cdots \times a_n) \circ \omega^* = \lambda_n \circ \pi^* \circ (a_1 \times \cdots \times a_n)$$

for some $\omega \neq \mathrm{id}_n$. If we compose on the right with $(\pi^{-1})^*$, write $b_i = a_{\pi i}$ and $\rho = \pi^{-1} \circ \omega^{-1} \circ \pi$ and pass all the permutations back through by (2.4), we find

$$\lambda_n \circ (\rho^{-1})^* \circ (b_{\rho 1} \times b_{\rho 2} \times \cdots \times b_{\rho n}) = \lambda_n \circ (b_1 \times b_2 \times \cdots \times b_n).$$

From Theorem 5.5, $b_{\rho j} = b_j$ for all j. In particular, let i be the smallest integer such that $\rho i \neq i$ and take $j = \rho^{-1} i$. Then Theorem 5.5 yields $b_i \perp b_j = b_{\rho j} = b_i$.

6. Loop spaces

We give two applications of Section 5 to loop spaces. The first concerns loop space functors, and the second is the proof of Theorem 1.2.

The usual space ΩX of loops on a based space X is not a monoid, but preserves products, $\Omega(X \times Y) = \Omega X \times \Omega Y$. On the other hand, Moore's loop space $\Omega_M X$ (e.g. [1, Section 7] or [2, p. 93]) is a monoid, but fails to preserve products. Generally, any functor L for which LX is always homotopy equivalent to ΩX is called a loop space functor. A weaker form of the following result was discussed in [2].

6.1. Proposition. *No loop space functor can preserve products and take H-spaces with strict identity as values.*

Proof. Suppose L is a functor such that LY is always an H-space with strict identity e, that is, $ey = y = ye$ for all $y \in LY$. There is a PROP $\mathscr{P}$, say, that acts on such H-spaces: it has one constant e, one unary operation id, and a nonassociative noncommutative multiplication as binary operation having e as strict identity.

If L also preserves products, then for any space X, $L\Omega_M X$ will be a $\mathscr{M}^\# \otimes \mathscr{P}$-space. But in $\mathscr{P}(1,1)$, $\mathrm{id} \perp \mathrm{id}$, so that $\mathscr{M}^\# \otimes \mathscr{P} = \mathscr{C}\mathscr{M}$ from Theorem 5.5. Therefore $L\Omega_M X$ is a commutative monoid, and (as in [1]) certainly not in general homotopy equivalent to $\Omega^2 X$.

Proof of Theorem 1.2. We rely heavily on the constructions of [2]. We recall that an E-space is a topological space X together with an action of some PROP $\mathscr{E}$ on X whose morphism spaces $\mathscr{E}(n,1)$ are contractible. The little cube PROP $\mathscr{Q} = \mathscr{Q}_\infty$ of [2, (2.49)] is a suitable example. We assume that the E-space X admits a homotopy inverse, which for example is the case when $\pi_0(X)$ is a group and X is numerably contractible (see [3, (12.7)]). We intend to apply the lifting theorem [2, (3.17(ii))] for the universal construction W to the diagram of PROPs

$$\begin{tikzcd} W(\mathcal{M}^{\#} \otimes \mathcal{Q}) \arrow[r, dashed] \arrow[d, "\varepsilon"] & \mathcal{E} \arrow[d] \\ \mathcal{M}^{\#} \otimes \mathcal{Q} \arrow[r] & \mathcal{C}\mathcal{M} \end{tikzcd}$$

where ε is the augmentation functor. We need to know that the spaces $(\mathcal{M}^{\#} \otimes \mathcal{Q})$ $(n, 1)$ are $\mathcal{S}_n$-free (which is true by Corollary 5.7) and paracompact. The latter follows from Theorem 5.5, since $\mathcal{Q}(1, 1)^n$ is paracompact and $(\mathcal{M}^{\#} \otimes \mathcal{Q})(n, 1)$ is obtained from $\mathcal{S}_n \times \mathcal{Q}(1, 1)^n$ by identifying the various copies $\{\pi\} \times \mathcal{Q}(1, 1)^n$ along closed subspaces of $\mathcal{Q}(1, 1)^n$. It follows that X admits a $W(\mathcal{M}^{\#} \otimes \mathcal{Q})$-structure, which is induced (canonically up to homotopy) by the given E-structure.

Applying the construction M [2, (4.49)] we obtain a $\mathcal{M}^{\#} \otimes \mathcal{Q}$-space Y and a homotopy $(\mathcal{M}^{\#} \otimes \mathcal{Q})$-homotopy equivalence $X \simeq Y$ in the sense of [2, (4.1)]. In particular, Y is a monoid, and also admits a homotopy inverse. Its classifying space BY inherits a $\mathcal{Q}$-structure from Y and the canonical map $Y \to \Omega BY$ is a $\mathcal{M}^{\#}$-homotopy equivalence in the category of $\mathcal{Q}$-spaces by [2, (6.10)] and [9]. Hence the composite $X \simeq Y \simeq \Omega BY$ is a homotopy $\mathcal{Q}$-delooping of X, and the induction can proceed.

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